

IRREDUCIBLE COMPONENTS OF THE SPACE OF FOLIATIONS BY SURFACES

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ABSTRACT. Let \mathcal{F} be written as $f^*(\mathcal{G})$, where \mathcal{G} is a 1-dimensional foliation on \mathbb{P}^{n-1} and $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ a non-linear generic rational map. We use local stability results of singular holomorphic foliations, to prove that: if $n \geq 4$, a foliation \mathcal{F} by complex surfaces on \mathbb{P}^n is globally stable under holomorphic deformations. As a consequence, we obtain irreducible components for the space of two-dimensional foliations in \mathbb{P}^n . We present also a result which characterizes holomorphic foliations on \mathbb{P}^n , $n \geq 4$ which can be obtained as a pull back of 1- foliations in \mathbb{P}^{n-1} of degree $d \geq 2$.

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1. INTRODUCTION

A two singular foliation \mathcal{F} of a holomorphic manifold M , $\dim_{\mathbb{C}} \geq 3$, may be defined by:

- (1) A covering $\mathcal{U} = (U_{\alpha})_{\alpha \in A}$ of M by open sets.
- (2) A collection $(\eta_{\alpha})_{\alpha \in A}$ of integrable $(n-2)$ -forms, $\eta_{\alpha} \in \Omega^{n-2}(U_{\alpha})$, where $\eta_{\alpha} \neq 0$ and defines a 2-dimensional foliation in U_{α} .
- (3) A multiplicative cocycle $G := (g_{\alpha\beta})_{U_{\alpha} \cap U_{\beta} \neq \emptyset}$ such that $\eta_{\alpha} = g_{\alpha\beta} \eta_{\beta}$.

If $N_{\mathcal{F}}$ denotes the holomorphic line bundle represented by the cocycle G , the family $(\eta_{\alpha})_{\alpha \in A}$, defines a holomorphic section of the vector bundle $\Omega^{n-2}(M) \otimes N_{\mathcal{F}}$ i.e. an element η of the cohomology vector space $H^0(M, \Omega^{n-2}(M) \otimes N_{\mathcal{F}})$. The analytic subset $Sing(\eta) := \{p \in M | \eta(p) = 0\}$ is the singular set of \mathcal{F} . In the case of $M = \mathbb{P}^n$, the n -dimensional complex projective space, we have a theorem of Chow-type. Denote by $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ the natural projection, and consider $\pi^* \mathcal{F}$ of the foliation \mathcal{F} by π ; with the previous notations, $\pi^* \mathcal{F}$ is defined by $(n-2)$ -forms, $\pi^* \eta_{\alpha} \in \Omega^{n-2}[\pi^{-1}(U_{\alpha})]$. Recall that for $n \geq 2$ we have $H^1(\mathbb{C}^{n+1} \setminus \{0\}, \mathcal{O}^*) = \{1\}$: it is a result from Cartan. As a consequence, there exists a global holomorphic $(n-2)$ -form η on $\mathbb{C}^{n+1} \setminus \{0\}$ which defines $\pi^* \mathcal{F}$ on $\mathbb{C}^{n+1} \setminus \{0\}$. By Hartog's extension

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theorem, η can be extended holomorphically at 0. By construction we have $i_R\eta = 0$, where R is the radial vector field. This fact and the integrability condition imply that each coefficient of η is a homogeneous polynomial of degree $\deg(\mathcal{F}) + 1$. Moreover, if we take a section by a generic immersion of hyperplane $H := (i : \mathbb{P}^{n-1} \rightarrow \mathbb{P}^n)$, this procedure gives a foliation by curves $i^*(\mathcal{F})$ on \mathbb{P}^{n-1} . We then define the degree of \mathcal{F} , for short $\deg(\mathcal{F})$, as the degree of a generic section as before. From now on we will always assume that the singular set of \mathcal{F} has codimension greater or equal than two. The projectivisation of the set of $n-2$ -forms which satisfy the previous conditions will be denoted by $\mathbb{Fol}(d; 2, n)$, the space of 2-dimensional foliations on \mathbb{P}^n of degree d . Note that $\mathbb{Fol}(d; 2, n)$ can be considered as a quasi projective algebraic subset of $\mathbb{P}H^0(\mathbb{P}^n, \Omega^{n-2}(\mathbb{P}^n) \otimes \mathcal{O}_{\mathbb{P}^n}(d+n-1))$. In this scenario we have the following:

Problem: *Describe and classify the irreducible components of $\mathbb{Fol}(d; 2, n)$ on \mathbb{P}^n , such that $n \geq 3$.*

We observe that the classification of the irreducible components of $\mathbb{Fol}(0; 2, n)$ is given in [2, Th. 3.8 p. 46] and that the classification of the irreducible components of $\mathbb{Fol}(1; 2, n)$ is given in [17, Th. 6.2 and Cor. 6.3 p. 935-936]. We refer the reader to [2] and [17] and references therein for a detailed description of them. In the case of foliations of codimension 1, the definitions of foliation and degree are analogous and we denote by $\mathbb{Fol}(k, n)$ the space of codimension 1 foliations of degree k on \mathbb{P}^n , such that $n \geq 3$. The study of irreducible components of these spaces has been initiated by Jouanolou in [10], where the irreducible components of $\mathbb{Fol}(k, n)$ for $k = 0$ and $k = 1$ are described. In the case of codimension one foliations one can exhibit some kind of list of irreducible components in every degree, but this list is incomplete. In the paper [3], the authors proved that $\mathbb{Fol}(2, n)$ has six irreducible components, which can be described by geometric and dynamic properties of a generic element. We refer the reader to [3] and [11] for a detailed description of them. There are known families of irreducible components in which the typical element is a pull-back of a foliation on \mathbb{P}^2 by a rational map. Given a generic rational map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ of degree $\nu \geq 1$, it can be written in homogeneous coordinates as $f = (F_0, F_1, F_2)$ where F_0, F_1 and F_2 are homogeneous polynomials of degree ν . Now consider a foliation \mathcal{G} on \mathbb{P}^2 of degree $d \geq 2$. We can associate to the pair (f, \mathcal{G}) the pull-back foliation $\mathcal{F} = f^*\mathcal{G}$. The degree of the foliation \mathcal{F} is $\nu(d+2) - 2$ as proved in [4]. Denote by $PB(d, \nu; n)$ the closure in $\mathbb{Fol}(\nu(d+2) - 2, n)$, $n \geq 3$ of the set of foliations \mathcal{F} of the form $f^*\mathcal{G}$. Since $(f, \mathcal{G}) \rightarrow f^*\mathcal{G}$ is an algebraic parametrization of $PB(d, \nu; n)$ it follows that $PB(d, \nu; n)$ is an unirational irreducible algebraic subset of $\mathbb{Fol}(\nu(d+2) - 2, n)$, $n \geq 3$. We have the following result:

Theorem 1.1. *$PB(d, \nu; n)$ is a unirational irreducible component of $\mathbb{Fol}(\nu(d+2) - 2, n)$; $n \geq 3$, $\nu \geq 1$ and $d \geq 2$.*

The case $\nu = 1$, of linear pull-backs, was proven in [1], whereas the case $\nu > 1$, of nonlinear pull-backs, was proved in [4]. The search for new components of pull-back type for the space of codimension 1 foliations was considered in the Ph.D thesis of the author [6] and after in [7]. There we investigated branched rational maps and foliations with algebraic invariant sets of positive dimensions.

Recently, A.Lins Neto in [14] generalized the results contained in [12] about singularities of integrable 1-forms for the 2-dimensional case and he has obtained as a corollary components of linear pull-back type for the case of 2-dimensional

foliations on \mathbb{P}^n . In the present work we will explore the result contained in [14] and some ideas contained in [4] to show that, in fact, there exist families of irreducible components of non-linear pull-back type for the 2-dimensional case. We would like to mention that in [4] the authors have shown that linear pull-back components exist in all codimension. However, the techniques that they use to prove this fact can not be applied to the non-linear case.

1.1. The present work. Let us describe, briefly, the type of pull-back foliation that we shall consider.

Let us fix some homogeneous coordinates $Z = (z_0, \dots, z_n)$ on \mathbb{C}^{n+1} and $X = (x_0, \dots, x_{n-1})$ on \mathbb{C}^n . Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ be a rational map represented in the coordinates $Z \in \mathbb{C}^{n+1}$ and $X \in \mathbb{C}^n$ by $\tilde{f} = (F_0, F_1, \dots, F_{n-1})$ where $F_i \in \mathbb{C}[X]$ are homogeneous polynomials without common factors of degree ν . Let \mathcal{G} be a foliation by curves on \mathbb{P}^{n-1} . This foliation can be represented in these coordinates by a homogenous polynomial $(n-2)$ -form of the type

$$\Omega = (-1)^{i+k+1} \sum_{i,k} x_k P_i dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_{n-1}$$

for all $i, k \in \{0, \dots, n-1\}$ where each P_i is a homogeneous polynomial of degree d . The pull back foliation $f^*(\mathcal{G})$ is then defined in homogeneous coordinates by the $(n-2)$ -form

$$\tilde{\eta}_{[f, \mathcal{G}]}(Z) = \left[(-1)^{i+k+1} \sum_{i,k} F_k (P_i \circ \tilde{f}) dF_0 \wedge \dots \wedge \widehat{dF_i} \wedge \dots \wedge \widehat{dF_k} \wedge \dots \wedge dF_{n-1} \right],$$

$i, k \in \{0, \dots, n-1\}$ where each coefficient of $\tilde{\eta}_{[f, \mathcal{G}]}(W)$ has degree $\Theta(\nu, d, n) + 1 = [(d+n-1)\nu - 2]$. Let $PB(\Theta, 2, n)$ be the closure in $\mathbb{F}ol(\Theta; 2, n)$ of the set $\{\tilde{\eta}_{[f, \mathcal{G}]}\}$, where $\tilde{\eta}_{[f, \mathcal{G}]}$ is as before. The pull-back foliation's degree is $\Theta(\nu, d, n) = [(d+n-1)\nu - 3]$ and for simplicity we will denote it by Θ . Let us state the main result of this work.

Theorem A. *$PB(\Theta; 2, n)$ is a unirational irreducible component of $\mathbb{F}ol(\Theta; 2, n)$ for all $n \geq 4$, $\nu \geq 2$ and $d \geq 2$.*

It is worth pointing out that the case $n = 3$ is also true and it is contained in theorem 1.1. So we can think this result as the $n \geq 4$ -dimensional generalization of [4] for 2-dimensionsoal foliations in \mathbb{P}^n

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2. 1-DIMENSIONAL FOLIATIONS ON \mathbb{P}^{n-1}

2.1. Basic facts. We recall the basic definitions and properties of foliations by curves on \mathbb{P}^{n-1} that we will use in this work. Proofs and details can be found in [16] and [14].

Let $R = \sum_{i=0}^{n-1} x_i \frac{\partial}{\partial x_i}$ be the radial vector field in \mathbb{C}^n . Denote by $\Sigma(R, d-1) = \{\mathcal{Z} | [R, \mathcal{Z}] = (d-1)\mathcal{Z}\}$, where $[R, \mathcal{Z}]$ stands for the Lie's bracket between the two vector fields R and \mathcal{Z} . Observe that $\Sigma(R, d-1)$ is a finite dimensional vector space whose elements are homogeneous polynomials of degree d . Let us write $\mathcal{X} =$

$(\mathcal{X}_0, \dots, \mathcal{X}_n)$ and $\nabla \mathcal{X} = \sum_{i=0}^{n-1} \frac{\partial \mathcal{X}}{\partial x_i}$. Let $\mathcal{E}(R, d-1) = \{\mathcal{X} \in \Sigma(R, d-1) | \nabla \mathcal{X} = 0\}$, and $\mathcal{K}(R, d-1) = \{\mathcal{X} \in \mathcal{E}(R, d-1) | \mathcal{X} \text{ has an isolated singularity at } 0 \in \mathbb{C}^n\}$. It can be verified that $\mathcal{K}(R, d-1)$ is a Zariski open and dense subset of $\Sigma(R, d-1)$ and for each $\mathcal{X} \in \mathcal{K}(R, d-1)$ then the $(n-2)$ -form

$$\Omega = i_R i_{\mathcal{X}} d\sigma = (-1)^{i+k+1} \sum_{i,k} x_k P_i dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_{n-1},$$

where $d\sigma = dx_0 \wedge \dots \wedge dx_i \wedge \dots \wedge dx_k \wedge \dots \wedge dx_{n-1}$ is the volume form in \mathbb{C}^n , $\mathcal{X} = \sum_i P_i \frac{\partial}{\partial x_i}$ and $0 \in \text{Sing}(\Omega)$ is a n.g.K singularity, with rotational $(d+n)\mathcal{X}$ (see section 5.2) and [14] for more details. Observe that if $\text{cod } \text{Sing}(\Omega) \geq 2$ then Ω defines a 1-dimensional foliation \mathcal{G} on \mathbb{P}^{n-1} of degree d .

Definition 2.1. Let us denote by $\text{Fol}(d; 1, n-1)$ the set of 1-dimensional foliations on \mathbb{P}^{n-1} .

Theorem 2.2. [16] *Given, $n \geq 3$, and $d \geq 2$ there exists a Zariski open subset $\mathcal{M}(d)$ of $\text{Fol}(d; 1, n-1)$ such that any \mathcal{G} satisfies:*

- (1) \mathcal{G} has exactly $N = \frac{d^n-1}{d-1}$ hyperbolic singularities and is regular on the complement.
- (2) \mathcal{G} has no invariant algebraic curve.

Let X be a germ of vector field at $0 \in \mathbb{C}^{n-1}$ with an isolated singularity at 0 and denote by $\lambda_1, \dots, \lambda_{n-1} \in \mathbb{C}$ the spectrum of its linear part. We say that X is hyperbolic at 0 if none of the quotients $\frac{\lambda_i}{\lambda_j}$ are real. We have the following proposition:

Proposition 2.3. *Let Q be a germ of vector field with a hyperbolic singularity at $0 \in \mathbb{C}^{n-1}$ and denote by $\lambda_1, \dots, \lambda_{n-1} \in \mathbb{C}$ its spectrum. Then, there are exactly $n-1$ germs of irreducible invariant analytic invariant curves $\Gamma_1, \Gamma_2, \dots, \Gamma_{n-1}$ at 0 where each Γ_i is smooth and tangent to the eigendirection corresponding to λ_i .*

In a local coordinate system near the singularity for instance, $0 \in (\mathbb{C}^{n-1}, u)$ where $u = (u_1, \dots, u_{n-1})$ the foliation can be written as

$$Q(u) = (\lambda_1 u_1) \frac{\partial}{\partial u_1} + \dots + (\lambda_{n-1} u_{n-1}) \frac{\partial}{\partial u_{n-1}} + h.o.t,$$

where *h.o.t* stands for higher order terms. Let us denote by $\mathbb{PK}(R, d-1) = L(d)$ and let $A(d) := \mathcal{M}(d) \cap L(d)$ be their intersection. An element of the open and dense subset $A(d) \subset \text{Fol}(d; 1, n-1)$ is the well-known generalized Joanoulou's example, see [16] and [14].

3. RATIONAL MAPS

Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ be a rational map and $\tilde{f} : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^n$ its natural lifting in homogeneous coordinates. We are considering the same homogeneous coordinates used in the introduction.

The *indeterminacy locus* of f is, by definition, the set $I(f) = \Pi_n(\tilde{f}^{-1}(0))$. Observe that the restriction $f|_{\mathbb{P}^n \setminus I(f)}$ is holomorphic. We characterize the set of rational maps used throughout this text as follows:

Definition 3.1. We denote by $RM(n, n-1, \nu)$ the set of maps $\{f : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}\}$ of degree ν given by $f = (F_0 : F_1 : \dots : F_{n-1})$ where the F_{j_s} , are homogeneous polynomials without common factors, with the same degree.

Let us note that the indeterminacy locus $I(f)$ is the intersection of the hypersurfaces $\Pi_n(F_i = 0)$ and $\Pi_n(F_j = 0)$ for $i \neq j$.

Definition 3.2. We say that $f \in RM(n, n-1, \nu)$ is *generic* if for all $p \in \tilde{f}^{-1}(0) \setminus \{0\}$ we have $dF_0(p) \wedge dF_1(p) \wedge \dots \wedge dF_{n-1}(p) \neq 0$.

This is equivalent to saying that $f \in RM(n, n-1, \nu)$ is *generic* if $I(f)$ is the transverse intersection of the n hypersurfaces $\Pi_n(F_i = 0)$ for $i = 0, \dots, n-1$. Moreover if f is generic and $\deg(f) = \nu$, then by Bezout's theorem $I(f)$ consists of ν^n distinct points.

Now let $V(f) = \mathbb{P}^n \setminus I(f)$, $P(f)$ the set of critical points of f in $V(f)$ and $C(f) = f(P(f))$ the set of the critical values of f . If f is generic, then $\overline{P(f)} \cap I(f) = \emptyset$, so that $\overline{P(f)} = P(f) \subset V(f)$ (where \overline{A} denotes the closure of $A \subset \mathbb{P}^n$ in the usual topology). Since $P(f) = \{p \in V(f); \text{rank}(df(p)) \leq (n-2)\}$, it follows from Sard's theorem that $C(f) = f(P(f))$ is a subset of Lebesgue's measure 0 in \mathbb{P}^{n-1} , in fact $C(f)$ is an algebraic curve.

The set of generic maps will be denoted by $Gen(n, n-1, \nu)$. We state the following result, whose proof is standard in algebraic geometry:

Proposition 3.3. *Gen(n, n-1, \nu) is a Zariski dense subset of RM(n, n-1, \nu).*

4. GENERIC PULL-BACK COMPONENTS - GENERIC CONDITIONS

Definition 4.1. Let $f \in Gen(n, n-1, \nu)$. We say that $\mathcal{G} \in A(d)$ is in generic position with respect to f if $Sing(\mathcal{G}) \cap C(f) = \emptyset$.

In this case we say that (f, \mathcal{G}) is a generic pair. In particular, when we fix a map $f \in Gen(n, n-1, \nu)$ the set $\mathcal{A} = \{\mathcal{G} \in A(d) | Sing(\mathcal{G}) \cap C(f) = \emptyset\}$ is an open and dense subset in $A(d)$ [15], since $C(f)$ is an algebraic curve in \mathbb{P}^{n-1} . The set $U_1 := \{(f, \mathcal{G}) \in Gen(n, n-1, \nu) \times A(d) | Sing(\mathcal{G}) \cap C(f) = \emptyset\}$ is an open and dense subset of $Gen(n, n-1, \nu) \times A(d)$. Hence the set $\mathcal{W} := \{\tilde{\eta}_{[f, \mathcal{G}]} | (f, \mathcal{G}) \in U_1\}$ is an open and dense subset of $PB(\Theta; 2, n)$.

Consider the set of foliations $\mathbb{Fol}(d; 1, n-1)$, $d \geq 2$, and the following map:

$$\begin{aligned} \Phi : RM(n, n-1, \nu) \times \mathbb{Fol}(d; 1, n-1) &\rightarrow \mathbb{Fol}(\Theta; 2, n) \\ (f, \mathcal{G}) &\rightarrow f^*(\mathcal{G}) = \Phi(f, \mathcal{G}). \end{aligned}$$

The image of Φ can be written as:

$$\left[(-1)^{i+k+1} \sum_{i,k} F_k(P_i \circ \tilde{f}) dF_0 \wedge \dots \wedge \widehat{dF_i} \wedge \dots \wedge \widehat{dF_k} \wedge \dots \wedge dF_{n-1} \right],$$

$i, k \in \{0, \dots, n-1\}$. Recall that $\Phi(f, \mathcal{G}) = \tilde{\eta}_{[f, \mathcal{G}]}$. More precisely, let $PB(\Theta, 2, n)$ be the closure in $\mathbb{Fol}(\Theta; 2, n)$ of the set of foliations \mathcal{F} of the form $f^*(\mathcal{G})$, where $f \in RM(n, n-1, \nu)$ and $\mathcal{G} \in \mathbb{Fol}(d; 1, n-1)$. Since $RM(n, n-1, \nu)$ and $\mathbb{Fol}(d; 1, n-1)$ are irreducible algebraic sets and the map $(f, \mathcal{G}) \rightarrow f^*(\mathcal{G}) \in \mathbb{Fol}(\Theta; 2, n)$ is an algebraic parametrization of $PB(\Theta, 2, n)$, we have that $PB(\Theta, 2, n)$ is an irreducible algebraic subset of $\mathbb{Fol}(\Theta; 2, n)$. Moreover, the set of generic pull-back foliations $\{\mathcal{F}; \mathcal{F} = f^*(\mathcal{G}), \text{ where } (f, \mathcal{G}) \text{ is a generic pair}\}$ is an open (not Zariski) and dense subset of $PB(\Theta, 2, n)$ for $\nu \geq 2, d \geq 2$.

Remark 4.2. We observe that if $\nu = 1$ the theorem is also true and, in this case, we re-obtain the result [14, Cor. 1 p.7] and [4, Cor. 5.1 p. 426] for the case of bi-dimensional foliations.

Remark 4.3. *To visualize that the degree of a generic pull-back foliation is indeed $\Theta(\nu, d, n) = [(d+n-1)\nu - 3]$, do the pull-back of a generic map of the Joanoulou's foliation on \mathbb{P}^{n-1} to obtain that the degree of this generic element coincides with this number.*

5. DESCRIPTION OF GENERIC PULL-BACK FOLIATIONS ON \mathbb{P}^n

5.1. The Kupka set of $\mathcal{F} = f^*(\mathcal{G})$. Let q_i be a singularity of \mathcal{G} and $V_{q_i} = \overline{f^{-1}(q_i)}$. If (f, \mathcal{G}) is a generic pair then $V_{q_i} \setminus I(f)$ is contained in the Kupka set of \mathcal{F} .

Fix $p \in V_{q_i} \setminus I(f)$. Since f is a submersion at p , there exist local analytic coordinate systems $(U, y, t), y : U \rightarrow \mathbb{C}^{n-1}, t : U \rightarrow \mathbb{C}$, and $(V, u), u : V \rightarrow \mathbb{C}^{n-1}$, at p and $q_i = f(p)$ respectively, such that $f(y_1, y_2, \dots, y_{n-1}, t) = (y_1, y_2, \dots, y_{n-1})$, $u(q) = 0$. Suppose that \mathcal{G} is represented by the vector field $Q = \sum_{j=1}^{n-1} Q_j(u) \frac{\partial}{\partial u_j}$ in a neighborhood of q_i . Then \mathcal{F} is represented by $Y = \sum_{j=1}^{n-1} Y_j(y) \frac{\partial}{\partial y_j}$. It follows that in U , the foliation \mathcal{F} is equivalent to the product of two foliations of dimension one: the singular foliation induced by the vector field Y in $(\mathbb{C}^{n-1}, 0)$ and the regular foliation of dimension one given by the fibers of the first projection $F(y, t) = y$. Note that the curve $\gamma := \{(y, t) | y = 0\}$ is contained in the singular set of \mathcal{F} . Moreover, the condition $\mathcal{G} \in A(d)$ implies that $\text{Div}(Y(p)) \neq 0$. This is the Kupka-Reeb phenomenon, and we have that p is in the Kupka set of \mathcal{F} (see [14]). It is known that this local product structure is stable under deformations of \mathcal{F} . Therefore if p is as before it belongs to the Kupka-set of \mathcal{F} . It is known that this local product structure is stable under small perturbations of \mathcal{F} [9],[18].

Remark 5.1. *Note that, \mathcal{F} has other singularities which are contained in $f^{-1}(C(f))$. We remark that \mathcal{F} has local holomorphic first integral in a neighborhood of each singularity of this type. In fact, this is the obstruction to try to apply the results contained in [4] to prove theorem A since these pull-back foliations do not have totally decomposable tangent sheaf.*

Since \mathcal{G} has degree d and all of its singularities are non degenerate it has $N = \frac{d^n-1}{d-1}$ singularities, say, q_1, \dots, q_N . We will denote the curves $f^{-1}(q_1), \dots, f^{-1}(q_N)$ by V_{q_1}, \dots, V_{q_N} respectively. We have the following:

Proposition 5.2. *For each $\{j = 1, \dots, N\}$, V_{q_j} is a complete intersection of $(n-1)$ transversal algebraic hypersurfaces. Furthermore, $V_{q_j} \setminus I(f)$ is contained in the Kupka set of $\mathcal{F} = f^*\mathcal{G}$.*

5.2. Generalized Kupka singularities for 2-dimensional foliations. In this section we will recall the generalized Kupka singularities of an integrable holomorphic $(n-2)$ -form, for more detail we refer the reader to [14]. They appear in the indeterminacy set of f and play a central role in great part of the proof of the main theorem. Let Ω be an holomorphic integrable $(n-2)$ -form defined in a neighborhood of $p \in \mathbb{C}^n$. In particular, since $d\Omega$ is a $(n-1)$ -form, there exists a holomorphic vector field \mathcal{Z} defined in a neighborhood of p such that: $d\Omega = i_{\mathcal{Z}} dw_0 \wedge \dots \wedge dw_{n-1}$.

Definition 5.3. We say that p is a singularity of generalized Kupka type of Ω if $\mathcal{Z}(p) = 0$ and p is an isolated zero of \mathcal{Z} .

Definition 5.4. We say that p is a nilpotent generalized singularity, for short n.g.k singularity, if the linear part of \mathcal{Z} , $D\mathcal{Z}(p)$ is nilpotent.

This definition is justified by the following result (that can be found in [14]).

Theorem 5.5. *Assume that $0 \in \mathbb{C}^n$ is a n.g.k singularity of Ω . Then there exists two holomorphic vector fields S and Z and a holomorphic coordinate system $x = (x_0, \dots, x_{n-1})$ around $0 \in \mathbb{C}^n$ where Ω has polynomial coefficients. More precisely, there exists two polynomial vector fields X and Y in \mathbb{C}^n such that:*

- (a) $Y = S + N$, where $S = \sum_{j=0}^{n-1} k_j w_j \frac{\partial}{\partial w_j}$ is linear semi-simple with eigenvalues $k_0, \dots, k_{n-1} \in \mathbb{N}$, $DN(0)$ is linear nilpotent and $[S, N] = 0$.
- (b) $[N, X] = 0$ and $[S, X] = kX$, where $k \in \mathbb{N}$. In other words, X is quasi-homogeneous with respect to S with weight k .
- (c) In this coordinate system we have $\Omega = i_Y i_X dx_0 \wedge \dots \wedge dx_{n-1}$ and $L_Y(\Omega) = (k + \text{tr}(S))\Omega$.

In particular, the foliation given by $\Omega = 0$ can be defined by a local action of the affine group.

Definition 5.6. In the situation of the theorem 5.5, $S = \sum_{j=0}^{n-1} k_j x_j \frac{\partial}{\partial x_j}$ and $L_S(X) = kX$, we say that the n.g.K is of type $(k_0, \dots, k_{n-1}; k)$.

Remark 5.7. We would like to observe that in many cases it can be proved that the vector field N of the statement of theorem 2 vanishes. In order to discuss this assertion it is convenient to introduce some objects. Given two germs of vector fields Z and W set $L_Z(W) := [Z, W]$. Recall that $\Sigma(S, \ell) = \{Z \in X \mid L_S(Z) = \ell Z\}$. Let X and $Y = S + N$ be as in theorem 5.5. Observe that:

- Jacobi's identity implies that if $W \in \Sigma(S, k)$ and $Z \in \Sigma(S, \ell)$ then $[W, Z] \in \Sigma(S, k + \ell)$.
- For all $k \in \mathbb{Z}$ we have $\dim_{\mathbb{C}}(\Sigma(S, k)) < \infty$ (because $k_0, \dots, k_{n-1} \in \mathbb{N}$).
- $N \in \Sigma(S, 0)$, $X \in \Sigma(S, \ell)$ and $L_X(N) = 0$, so that $N \in \ker(L_X^0)$, where $L_X^0 := LX : \Sigma(S, 0) \rightarrow \Sigma(S, \ell)$. In particular, the vector field $N \in \Sigma(S, 0)$ of theorem 5.5 necessarily vanishes $\iff \ker(L_X^0) = \{0\}$.

In [14], § 3.2 it is shown that under a non-resonance condition, which depends only on X , then $\ker(L_X^0) = \{0\}$. Let us mention some correlated facts.

- (I) If S has no resonances of the type $\langle \sigma, k \rangle - k_j = 0$, where $\langle \sigma, k \rangle = \sum_j \sigma_j \cdot k_j$, $k = (k_0, \dots, k_{n-1})$ and $\sigma = (\sigma_0, \dots, \sigma_{n-1}) \in \mathbb{Z}_{\geq 0}^n$, then $\ker(L_X^0) = \{0\}$.
- (II) When $n = 3$ and X has an isolated singularity at $0 \in \mathbb{C}^3$ then $\ker(L_X^0) = \{0\}$ (c.f [12]).
- (III) When $N \not\equiv 0$ and $\text{cod}_{\mathbb{C}}(\text{sing}(N)) = 1$, or $\text{sing}(N)$ has an irreducible component of dimension one then it can be proved that X cannot have an isolated singularity at $0 \in \mathbb{C}^n$.

In fact, we think that whenever X has an isolated singularity at $0 \in \mathbb{C}^n$ and $\nabla X = 0$ then $\ker(L_X^0) = \{0\}$.

The next result is about the nature of the set $\mathcal{K}(S, \ell) := \{X \in \Sigma(S, \ell) \mid \ker(L_X^0) = \{0\} \text{ and } \nabla X = 0\}$.

Proposition 5.8. *If $\mathcal{K}(S, \ell) \neq \emptyset$ then $\mathcal{K}(S, \ell)$ is a Zariski open and dense subset of $\mathcal{E}(S, \ell)$. In particular, if there exists $X \in \mathcal{E}(S, \ell)$ satisfying the non-resonance condition mentioned in remark 5.7 then $\mathcal{K}(S, \ell)$ is a Zariski open and dense in $\mathcal{E}(S, \ell)$. Proposition 5.8 is a straightforward consequence of the following facts:*

- (A) The set of linear maps $\mathcal{L}(\Sigma(S, 0), \Sigma(S, \ell))$ is finite dimensional vector space. Moreover, the subspace $\mathcal{NI} := \{T \in \mathcal{L}(\Sigma(S, 0), \Sigma(S, \ell)) \mid T \text{ is not injective}\}$ is an algebraic subset of $\mathcal{L}(\Sigma(S, 0), \Sigma(S, \ell))$.
- (B) The map $L : \mathcal{E}(S, \ell) \rightarrow \mathcal{L}(\Sigma(S, 0), \Sigma(S, \ell))$ defined by $L(X) = L_X^0$ is linear. As a consequence, the set $L^{-1}(\mathcal{NI})$ is an algebraic subset of $\mathcal{E}(S, \ell)$.
- (C) $\mathcal{K}(S, \ell) = \mathcal{E}(S, \ell) \setminus L^{-1}(\mathcal{NI})$.

We leave the details to the reader.

Remark 5.9. In the case of the radial vector field, $R = \sum_{i=0}^{n-1} x_i \frac{\partial}{\partial x_i}$, we have $\mathcal{K}(R, d-1) \neq \emptyset$ for all $d \geq 2$. In fact, it is proved in [14], § 3.2 that $J_d \in (R, d-1)$, where J_d is the generalized Jouanolou's vector field.

Consider a holomorphic family of $(n-2)$ -forms, $(\Omega_t)_{t \in U}$, defined on a polydisc Q of \mathbb{C}^n , where the space of parameters U is an open set of \mathbb{C}^k with $0 \in U$. Let us assume that:

- For each $t \in U$ the form Ω_t defines a 2-dimensional foliation \mathcal{F}_t on Q . Let $(Z_t)_{t \in U}$ be the family of holomorphic vector fields on Q such that $d\Omega_t = i_{Z_t} dx_0 \wedge \cdots \wedge dx_{n-1}$.
- \mathcal{F}_0 has a n.g.K singularity at $0 \in Q$.

We can now state the stability result, whose proof can be found in [14]:

Theorem 5.10. In the above situation there exists a neighborhood $0 \in V \subset U$, a polydisc $0 \in P \subset Q$, and a holomorphic map $\mathcal{P} : V \rightarrow P \subset \mathbb{C}^n$ such that $\mathcal{P}(0) = 0$ and for any $t \in V$ then $\mathcal{P}(t)$ is the unique singularity of \mathcal{F}_t in P . Moreover, $\mathcal{P}(t)$ is the same type as $\mathcal{P}(0)$ in the sense that: If 0 is a n.g.K singularity of type (k_0, \dots, k_{n-1}, k) of \mathcal{F}_0 then $\mathcal{P}(t)$ is a n.g.K singularity of type (k_0, \dots, k_{n-1}, k) of \mathcal{F}_t , $\forall t \in V$.

Let us now describe $\mathcal{F} = f^*(\mathcal{G})$ in a neighborhood of a point $p \in I(f)$.

Proposition 5.11. If $p \in I(f)$ then p is a n.g.K singularity of Ω of type $(1, \dots, 1, n)$.

Proof. It is easy to show that there exists a local chart $(U, x = (x_0, \dots, x_{n-1})) \in \mathbb{C}^n$ around p such that the lifting \tilde{f} of f is of the form $\tilde{f}|_U = (x_0, \dots, x_{n-1}) : U \rightarrow \mathbb{C}^n$. In particular $\mathcal{F}|_{U(p)}$ is represented by the homogeneous $(n-2)$ -form

$$\Omega = (-1)^{i+k+1} \sum_{i,k} x_k P_i dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_{n-1}.$$

Observe that $L_R \Omega = (d+n)\Omega$, $\mathcal{X} = \sum_i P_i \frac{\partial}{\partial x_i}$, $\mathcal{Z} = (d+n)\mathcal{X}$, $[R, \mathcal{X}] = d\mathcal{X}$. Since we are considering $\Omega \in \mathcal{A}$ we have that $Y = R$, $N = 0$ and hence we conclude that p is a n.g.K singularity of Ω of type $(1, \dots, 1, n)$. In particular the vector field S as in the Theorem 5.5 is the radial vector field. \square

It follows from theorem. 5.10 that this singularities are stable under deformations. Proposition 5.11 says that the germ $f^*\mathcal{G}, p$ of $f^*\mathcal{G}$ at $p \in I(f)$ is equivalent to the germ of $\Pi_{n-1}^*(\mathcal{G}), 0$, where $\Pi_{n-1} : \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1}$ is the canonical projection. Therefore in this case we can see all the foliations \mathcal{G} in a neighborhood of each $p \in I(f)$.

5.2.1. *Deformations of the singular set of $\mathcal{F}_0 = f_0^*(\mathcal{G}_0)$.* We have constructed an open and dense subset \mathcal{W} inside $PB(\Theta, 2, n)$ containing the generic pull-back foliations. We will show that for any rational foliation $\mathcal{F}_0 \in \mathcal{W}$ and any germ of a holomorphic family of foliations $(\mathcal{F}_t)_{t \in (\mathbb{C}, 0)}$ such that $\mathcal{F}_0 = \mathcal{F}_{t=0}$ we have $\mathcal{F}_t \in PB(\Theta, 2, n)$ for all $t \in (\mathbb{C}, 0)$.

Using the theorem. 5.10 with $V = (\mathbb{C}, 0)$, it follows that for each $p_j \in I(f_0)$ there exists a deformation $p_j(t)$ of p_j and a deformation of $\mathcal{F}_{t, p_j(t)}$ of \mathcal{F}_{p_j} such that $p_j(t)$ is a n.g.K singularity of $\mathcal{F}_{t, p_j(t)} := \Omega_{p_j(t)}$ of type $(1, \dots, 1, n)$ and $(\mathcal{F}_{t, p_j(t)})_{t \in (\mathbb{C}, 0)}$ defines a holomorphic family of foliations in \mathbb{P}^{n-1} . We will denote by $I(t) = \{p_1(t), \dots, p_j(t), \dots, p_{\nu^n}(t)\}$.

Remark 5.12. *Since $I(t)$ is not connected we can not guarantee a priori that $\mathcal{F}_{t, p_i(t)} = \mathcal{F}_{t, p_j(t)}$, if $i \neq j$.*

Lemma 5.13. *There exist $\epsilon > 0$ and smooth isotopies $\phi_{q_i} : D_\epsilon \times V_{q_i} \rightarrow \mathbb{P}^n, q_i \in \text{Sing}(\mathcal{G}_0)$, such that $V_{q_i}(t) = \phi_i(\{t\} \times V_{q_i})$ satisfies:*

- (a) $V_{q_i}(t)$ is an algebraic subvariety of dimension 1 of \mathbb{P}^n and $V_{q_i}(0) = V_{q_i}$ for all $q_i \in \text{Sing}(\mathcal{G}_0)$ and for all $t \in D_\epsilon$.
- (b) $I(t) \subset V_{q_i}(t)$ for all $q_i \in \text{Sing}(\mathcal{G}_0)$ and for all $t \in D_\epsilon$. Moreover, if $q_i \neq q_j$, and $q_i, q_j \in \text{Sing}(\mathcal{G}_0)$, we have $V_{q_i}(t) \cap V_{q_j}(t) = I(t)$ for all $t \in D_\epsilon$ and the intersection is transversal.
- (c) $V_{q_i}(t) \setminus I(t)$ is contained in the Kupka-set of \mathcal{F}_t for all $q_i \in \text{Sing}(\mathcal{G}_0)$ and for all $t \in D_\epsilon$. In particular, the transversal type of \mathcal{F}_t is constant along $V_{q_i}(t) \setminus I(t)$.

Proof. See [11, lema 2.3.3, p.83]. □

5.3. **End of the proof of Theorem A.** We divide the end of the proof of Theorem A in two parts. In the first part we construct a family of rational maps $f_t : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$, $f_t \in \text{Gen}(n, n-1, \nu)$, such that $(f_t)_{t \in D_\epsilon}$ is a deformation of f_0 and the subvarieties $V_{q_i}(t)$ are fibers of f_t for all t . In the second part we show that there exists a family of foliations $(\mathcal{G}_t)_{t \in D_\epsilon}, \mathcal{G}_t \in \mathcal{A}$ (see Section 4) such that $\mathcal{F}_t = f_t^*(\mathcal{G}_t)$ for all $t \in D_\epsilon$.

5.3.1. *Part 1.* Once $d = \deg(\mathcal{G}_0) \geq 2$, the number of singularities of \mathcal{G}_0 is $N = \frac{d^n - 1}{d - 1} > n$, so we can suppose that the singularities of \mathcal{G}_0 are $q_1 = [0 : 0 : \dots : 1], \dots, q_n = [1 : 0 : \dots : 0], \dots, q_N$.

Proposition 5.14. *Let $(\mathcal{F}_t)_{t \in D_\epsilon}$ be a deformation of $\mathcal{F}_0 = f_0^*(\mathcal{G}_0)$, where (f_0, \mathcal{G}_0) is a generic pair, with $\mathcal{G}_0 \in \mathcal{A}$, $f_0 \in \text{Gen}(n, n-1, \nu)$ and $\deg(f_0) = \nu \geq 2$. Then there exists a deformation $(f_t)_{t \in D_\epsilon}$ of f_0 in $\text{Gen}(n, n-1, \nu)$ such that:*

- (i) $V_{q_i}(t)$ are fibers of $(f_t)_{t \in D_{\epsilon'}}$.
- (ii) $I(t) = I(f_t), \forall t \in D_{\epsilon'}$.

Proof. Let $\tilde{f}_0 = (F_0, \dots, F_{n-1}) : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^n$ be the homogeneous expression of f_0 . Then V_{q_1}, V_{q_2}, \dots , and V_{q_n} appear as the complete intersections $V_{q_i} = (F_0 = F_1 = \dots = \widehat{F_{i-1}} = \dots = 0)$. The remaining fibers, V_{q_i} for $i > n$ are obtained in the same way. With this convention we have that $I(f_0) = V_{q_i} \cap V_{q_j}$ if $i \neq j$. It follows from [19] (see section 4.6 pp 235-236) that each $V_{q_i}(t)$ is also a smooth complete intersection generated by polynomials of the same degree. However we can not assure that the set of polynomials which define each $V_{q_i}(t)$ have a correlation among them. In

the next lines we will show this fact. For this let us work firstly with two curves. For instance, let us take $V_{q_1}(t)$ and $V_{q_2}(t)$ which are deformations of V_{q_1} and V_{q_2} respectively. We will see that this two curves are enough to construct the family of deformations $(f_t)_{t \in D_{\epsilon'}}$. After that we will show that the remaining curves $V_{q_i}(t)$ are also fibers of $(f_t)_{t \in D_{\epsilon'}}$. We can write $V_{q_1}(t) = (F_1(t) = F_2(t) = \dots F_{n-1}(t) = 0)$, and $V_{q_2}(t) = (\tilde{F}_0(t) = F_1(t) = \tilde{F}_2(t) \dots = \dots \tilde{F}_{n-1}(t) = 0)$ where $(F_i(t))_{t \in D_{\epsilon'}}$ and $(\tilde{F}_i(t))_{t \in D_{\epsilon'}}$ are deformations of F_i and $D_{\epsilon'}$ is a possibly smaller neighborhood of 0. Observe first that since the $F_{is}(t)$ and $\tilde{F}_{is}(t)$ are near F_{is} , they meet as a complete intersection at:

$$I(f_t) := (F_0(t) = 0) \cap V_1(t)$$

On the other hand we also have

$$I(t) = V_{q_1}(t) \cap V_{q_2}(t) = V_{q_1}(t) \cap [(F_0(t) = 0) \cap \{\tilde{F}_2(t) = \dots \tilde{F}_{n-1}(t) = 0\}].$$

Let us write $\{S(t) = 0\} = \{\tilde{F}_2(t) = \dots \tilde{F}_{n-1}(t) = 0\}$. Hence $I(f_t) \cap \{S(t) = 0\} = V_{q_1}(t) \cap V_{q_2}(t) = I(t)$, which implies that $I(t) \subset I(f_t)$. Since $I(f_t)$ and $I(t)$ have ν^n points, we have that $I(t) = I(f_t)$ for all $t \in D_{\epsilon'}$. In particular, we obtain that $I(t) \subset \{S(t) = 0\}$. We will use the following version of Noether's Normalization Theorem (see [11] p 86):

Lemma 5.15. (*Noether's Theorem*) *Let $G_0, \dots, G_k \in \mathbb{C}[z_1, \dots, z_m]$ be homogeneous polynomials where $0 \leq k \leq m$ and $m \geq 2$, and $X = (G_0 = \dots = G_k = 0)$. Suppose that the set $Y := \{p \in X \mid dG_0(p) \wedge \dots \wedge dG_k(p) = 0\}$ is either 0 or \emptyset . If $G \in \mathbb{C}[z_1, \dots, z_m]$ satisfies $G|_X \equiv 0$, then $G \in \langle G_0, \dots, G_k \rangle$.*

Take $k = n - 1$, $G_0 = F_0(t)$, $G_1 = F_1(t) \dots G_{n-1} = F_{n-1}(t)$. Using Noether's Theorem with $Y = 0$ and the fact that all polynomials involved are homogeneous of the same degree, we have $\tilde{F}_i(t) \in \langle F_0(t), F_1(t), \dots, F_{n-1}(t) \rangle$. More precisely we conclude that each $\tilde{F}_i(t) = \sum_{j=0}^{n-1} g_{i,j}(t) F_j(t)$, $g_{i,j}(t) \in \mathbb{C}$ and when $t = 0$ for each i , $\tilde{F}_i(0) = F_i(0) = F_i$. On the other hand, if $V_{q_k}(t)$ is the deformation of another V_{q_k} , then $V_{q_k}(t)$ is also a complete intersection, say, $V_{q_k}(t) = (P_1^k(t) = \dots = P_{n-1}^k(t) = 0)$ where each $P_i^k(t)$, for $i = 1, \dots, n - 1$ is a homogeneous polynomial of degree ν . Since $I(t) \subset (P_i^k(t) = 0)$ for $i = 1, \dots, n - 1$, we have that each $P_i^k(t)$ is a linear combination of the $F_{is}(t)$, that is, $P_i^k(t) \in \langle F_0(t), F_1(t), \dots, F_{n-1}(t) \rangle$. This implies that $V_{q_k}(t)$ is also a fiber of f_t , as the reader can check, say $V_{q_k}(t) = f_t^{-1}(q_k(t))$. Since $q_k(t) = f_t(V_{q_k}(t))$ and f_t and $V_{q_k}(t)$ are deformations of f_0 and V_{q_k} we get that $q_k(t)$ is a deformation of q_k , so that for small t , $q_k(t)$ is a regular value of f_t . \square

5.3.2. Part 2. Let us now define a family of foliations $(\mathcal{G}_t)_{t \in D_{\epsilon}}$, $\mathcal{G}_t \in \mathcal{A}$ (see Section 4) such that $\mathcal{F}_t = f_t^*(\mathcal{G}_t)$ for all $t \in D_{\epsilon}$. Let $M(t)$ be the family of "rational varieties" obtained from \mathbb{P}^n by blowing-up at the ν^n points $p_1(t), \dots, p_j(t), \dots, p_{\nu^n}(t)$ corresponding to $I(t)$ of \mathcal{F}_t ; and denote by

$$\pi(t) : M(t) \rightarrow \mathbb{P}^n$$

the blowing-up map. The exceptional divisor of $\pi(t)$ consists of ν^n submanifolds $E_j(t) = \pi(t)^{-1}(p_j(t))$, $1 \leq j \leq \nu^n$, which are projective spaces \mathbb{P}^{n-1} . More precisely, if we blow-up \mathcal{F}_t at the point $p_j(t)$, then the restriction of the strict transform $\pi^*\mathcal{F}_t$ to the exceptional divisor $E_j(t) = \mathbb{P}^{n-1}$ is up to a linear automorphism of \mathbb{P}^{n-1} , the homogeneous $(n - 2)$ -form that defines \mathcal{F}_t at the point $p_j(t)$. With

this process we produce a family of bidimensional holomorphic foliations in \mathcal{A} . This family is the “holomorphic path” of candidates to be a deformation of \mathcal{G}_0 . In fact, since \mathcal{A} is an open set we can suppose that this family is inside \mathcal{A} . We fix the exceptional divisor $E_1(t)$ to work with and we denote by \mathcal{G}_t the restriction of $\pi^*\mathcal{F}_t$ to $E_1(t)$. Consider the family of mappings $f_t : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$, $t \in D_{\epsilon'}$ defined in Proposition 5.14. We will consider the family $(f_t)_{t \in D_{\epsilon'}}$ as a family of rational maps $f_t : \mathbb{P}^n \dashrightarrow E_1(t)$; we decrease ϵ if necessary. We would like to observe that the mapping $f_t \circ \pi(t) : M(t) \setminus \cup_j E_j(t) \rightarrow \mathbb{P}^{n-1}$ extends as holomorphic mapping $\hat{f}_t : M(t) \rightarrow \mathbb{P}^{n-1}$ if $|t| < \epsilon$. This follows from the fact that $dF_0(t)(p_j(t)) \wedge dF_1(t)(p_j(t)) \wedge \dots \wedge dF_{n-1}(t)(p_j(t)) \neq 0$, $1 \leq j \leq \nu^n$, if $|t| < \epsilon$. The mapping f_t can be interpreted as follows. Each fiber of f_t meets $p_j(t)$ once, which implies that each fiber of \hat{f}_t cuts $E_1(t)$ only one time. Since $M(t) \setminus \cup_j E_j(t)$ is biholomorphic to $\mathbb{P}^n \setminus I(t)$, after identifying $E_1(t)$ with \mathbb{P}^{n-1} , we can imagine that if $q \in M(t) \setminus \cup_j E_j(t)$ then $\hat{f}_t(q)$ is the intersection point of the fiber $\hat{f}_t^{-1}(\hat{f}_t(q))$ with $E_1(t)$. We obtain a mapping

$$\hat{f}_t : M(t) \rightarrow \mathbb{P}^{n-1}.$$

With all these ingredients we can define the foliation $\tilde{\mathcal{F}}_t = f_t^*(\mathcal{G}_t) \in PB(\Theta, 2, n)$. This foliation is a deformation of \mathcal{F}_0 . Based on the previous discussion let us denote $\mathcal{F}_1(t) = \pi(t)^*(\mathcal{F}_t)$ and $\hat{\mathcal{F}}_1(t) = \pi(t)^*(\tilde{\mathcal{F}}_t)$.

Lemma 5.16. *If $\mathcal{F}_1(t)$ and $\hat{\mathcal{F}}_1(t)$ are the foliations defined previously, we have that*

$$\mathcal{F}_1(t)|_{E_1(t) \simeq \mathbb{P}^{n-1}} = \hat{\mathcal{G}}_t = \hat{\mathcal{F}}_1(t)|_{E_1(t) \simeq \mathbb{P}^{n-1}}$$

where $\hat{\mathcal{G}}_t$ is the foliation induced on $E_1(t) \simeq \mathbb{P}^{n-1}$ by the homogeneous $(n-2)$ -form $\Omega_{p_1(t)}$.

Proof. In a neighborhood of $p_1(t) \in I(t)$, \mathcal{F}_t is represented by the homogeneous $(n-2)$ -form $\Omega_{p_1(t)}$. This $(n-2)$ -form satisfies $i_{R(t)}\Omega_{p_1(t)} = 0$ and therefore naturally defines a foliation on \mathbb{P}^{n-1} . This proves the first equality. The second equality follows from the geometrical interpretation of the mapping $\hat{f}_t : M(t) \rightarrow \mathbb{P}^{n-1}$, since $\hat{\mathcal{F}}_1(t) = \hat{f}_t^*(\mathcal{G}_t)$. \square

Let $q_1(t)$ be a singularity of \mathcal{G}_t . Since the map $t \rightarrow q_1(t) \in \mathbb{P}^{n-1}$ is holomorphic, there exists a holomorphic family of automorphisms of \mathbb{P}^{n-1} , $t \rightarrow H(t)$ such that $q_1(t) = [0 : \dots : 1] \in E_1(t)$ is kept fixed. Observe that such a singularity has $(n-1)$ non algebraic separatrices at this point. Fix a local analytic coordinate system $(U_t = u_0(t), \dots, u_{n-1}(t))$ at $q_1(t)$ such that the local separatrices are tangents to $(u_i(t) = 0)$ for each i . Observe that the local smooth hypersurfaces along $\hat{V}_{q_1(t)} = \hat{f}_t^{-1}(q_1(t))$ defined by $\hat{U}_i(t) := (u_i(t) \circ \hat{f}_t = 0)$ are invariant for $\hat{\mathcal{F}}_1(t)$. Furthermore, they meet transversely along $\hat{V}_{q_1(t)}$. On the other hand, $\hat{V}_{q_1(t)}$ is also contained in the Kupka set of $\mathcal{F}_1(t)$. Therefore there are $(n-1)$ local smooth hypersurfaces $U_i(t) := (u_i(t) \circ \hat{f}_t = 0)$ invariant for $\mathcal{F}_1(t)$ such that:

- (1) All the $U_i(t)$ meet transversely along $\hat{V}_{q_1(t)}$.
- (2) $U_i(t) \cap \pi(t)^{-1}(p_1(t)) = (U_i(t) = 0) = \hat{U}_i(t) \cap \pi(t)^{-1}(p_1(t))$ (because $\mathcal{F}_1(t)$ and $\hat{\mathcal{F}}_1(t)$ coincide on $E_1(t) \simeq \mathbb{P}^{n-1}$).
- (3) Each $U_i(t)$ is a deformation of $U_i(0) = \hat{U}_i(0)$.

Choosing $i = 0$ we shall prove that $U_0(t) = \hat{U}_0(t)$ for small t . For our analysis this will be sufficient to finish the proof of Theorem A.

Lemma 5.17. $U_0(t) = \hat{U}_0(t)$ for small t .

Proof. Let us consider the projection $\hat{f}_t : M(t) \rightarrow \mathbb{P}^{n-1}$ on a neighborhood of the regular fibre $\hat{V}_{q_1(t)}$, and fix local coordinates $(U_t = u_0(t), \dots, u_{n-1}(t))$ on \mathbb{P}^{n-1} such that $U_1(t) := (u_1(t) \circ \hat{f}_t = 0)$. For small ϵ , let $H_\epsilon = (u_1(t) \circ \hat{f}_t = \epsilon)$. Thus $\hat{\Sigma}_\epsilon = \hat{U}_0(t) \cap H_\epsilon$ are (vertical) compact curves, deformations of $\hat{\Sigma}_0 = \hat{V}_{q_1(t)}$. Set $\Sigma_\epsilon = U_0(t) \cap \hat{H}_\epsilon$. The Σ'_ϵ s, as the $\hat{\Sigma}'_\epsilon$ s, are compact curves (for t and ϵ small), since $U_0(t)$ and $\hat{U}_0(t)$ are both deformations of the same U_0 . Thus for small t , $U_0(t)$ is close to $\hat{U}_0(t)$. It follows that $\hat{f}_t(\Sigma_\epsilon)$ is an analytic curve contained in a small neighborhood \tilde{U}_t of $q_1(t)$, for small ϵ . By the maximum principle, we must have that $\hat{f}_t(\Sigma_\epsilon)$ is a point, so that $\hat{f}_t(U_0(t)) = \hat{f}_t(\cup_\epsilon \Sigma_\epsilon)$ is a curve $C \subset \tilde{U}_t$, that is, $U_0(t) = \hat{f}_t^{-1}(C)$. But $U_0(t)$ and $\hat{U}_0(t)$ intersect the exceptional divisor $E_1(t) \simeq \mathbb{P}^{n-1}$ along the separatrix $(u_0(t) = 0)$ of \mathcal{G}_t through $q_1(t)$. This implies that $U_0(t) = \hat{f}_t^{-1}(C) = \hat{f}_t^{-1}(u_0(t) = 0) = \hat{U}_0(t)$. \square

We have proved that the foliations \mathcal{F}_t and $\hat{\mathcal{F}}_t$ have a common local leaf: the leaf that contains $\pi(t) \left(U_0(t) \setminus \hat{V}_{q_1(t)} \right)$ which is not algebraic. Let $D(t) := \text{Tang}(\mathcal{F}(t), \hat{\mathcal{F}}(t))$ be the set of tangencies between $\mathcal{F}(t)$ and $\hat{\mathcal{F}}(t)$. This set can be defined by $D(t) = \{Z \in \mathbb{C}^{n+1}; \Omega(t) \wedge \hat{\Omega}(t) = 0\}$, where $\Omega(t)$ and $\hat{\Omega}(t)$ define $\mathcal{F}(t)$ and $\hat{\mathcal{F}}(t)$, respectively. Hence it is an algebraic set. Since this set contains an immersed non-algebraic surface $U_0(t)$, we necessarily have that $D(t) = \mathbb{P}^n$. It follows that $\mathcal{F}_t = \hat{\mathcal{F}}_t$.

Recall from Definition 3.2 the concept of a generic map. Let $f \in RM((n, n-1, \nu), I(f))$ its indeterminacy locus and \mathcal{F} a foliation by complex surfaces on \mathbb{P}^n , $n \geq 4$. Consider the following properties:

\mathcal{P}_1 : Any point $p_j \in I(f)$ \mathcal{F} has the following local structure: there exists an analytic coordinate system (U^{p_j}, x^{p_j}) around p_j such that $x^{p_j}(p_j) = 0 \in (\mathbb{C}^n, 0)$ and $\mathcal{F}|_{(U^{p_j}, x^{p_j})}$ can be represented by a homogeneous $(n-2)$ -form Ω_{p_j} (as described in the Lemma 5.11) such that

- (a) $\text{Sing}(\mathcal{Z}_{p_j}) = 0$, where \mathcal{Z}_{p_j} is the rotational of Ω_{p_j} .
- (b) 0 is a n.g.K singularity of the type $(1, \dots, 1, n)$

\mathcal{P}_2 : There exists a fibre $f^{-1}(q) = V(q)$ such that $V(q) = f^{-1}(q) \setminus I(f)$ is contained in the Kupka-Set of \mathcal{F} .

\mathcal{P}_3 : $V(q)$ has transversal type Q , where Q is a germ of vector field on $(\mathbb{C}^{n-1}, 0)$ with at least a non algebraic separatrix and such that the Camacho-Sad index of \mathcal{G} with respect to this separatrix is non-real.

\mathcal{P}_4 : \mathcal{F} has no algebraic hypersurface.

Lemma 5.17 allows us to prove the following result:

Theorem B. *In the conditions above, if properties \mathcal{P}_1 , \mathcal{P}_2 , \mathcal{P}_3 and \mathcal{P}_4 hold then \mathcal{F} is a pull back foliation, $\mathcal{F} = f^*(\mathcal{G})$, where \mathcal{G} is a 1-dimensional foliation of degree $d \geq 2$ on \mathbb{P}^{n-1} .*

Note that the situation $n = 3$ is proved in [4, Th. B p. 709]. So we can think this result as $n \geq 4$ -dimensional generalization of [4] for bi-dimensional foliations in \mathbb{P}^n

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